WEAK SOLUTION OF THE SECOND ORDER EVOLUTION EQUATION WITH PARAMETER

by Mariusz Jużyniec

Abstract. The purpose of this paper is to present some theorems on the continuity and differentiability, with respect to the parameter h, of a weak solution of the evolution equation $\ddot{u}(t) = A(h)u(t) + f(h,t)$, in the case of operators $A(h)$ having domains depending on the parameter h.

1. Introduction. Let X be a Banach space. By X^* we denote its dual space. Let $B(X)$ be the totality of bounded linear operators. Function $\mathbb{R} \ni t \longrightarrow C(t) \in B(X)$ is called a strongly continuous cosine family in X if

$$\forall t, s \in \mathbb{R}: \; C(t+s) + C(t-s) = 2C(t)C(s), \tag{1}$$

$$C(0) = I, \tag{2}$$

$$\mathbb{R} \ni t \longrightarrow C(t)x \text{ is continuous for each fixed } x \in X. \tag{3}$$

The associated sine family is given by

$$S(t)x = \int_0^t C(r)xdr$$

for $x \in X$ and $t \in \mathbb{R}$. The infinitesimal generator is the operator $A : D(A) \longrightarrow X$ defined by $Ax = \lim_{h\to 0} 2h^{-2}(C(h) - I)x$ for $x \in D(A)$, where $D(A) = \{x \in X : \lim_{h\to 0} 2h^{-2}(C(h) - I)x \text{ exists}\}$.

The cosine family in X with generator A is associated with the Cauchy problem for the abstract evolution equation of second order in X

$$\frac{d^2u}{dt^2} = Au \quad t \in \mathbb{R}; \;\; u(0) = x, \;\; u'(0) = y. \tag{4}$$

For a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$ in X with the infinitesimal generator A, we define the set

$$E(A) = \{x \in X \; : \; C(\cdot)x \text{ is once continuously differentiable in } t \in \mathbb{R}\}.$$

For the reader's convenience and to establish notation we briefly recall the theory for such cosine and sine families.

THEOREM 1. ([**9**], Prop. 2.1.) *Let* $\{C(t)\}_{t\in\mathbb{R}}$ *be a cosine family and* $\{S(t)\}_{t\in\mathbb{R}}$ *be the associated sine family. Then*

(i) *for each* $t \in \mathbb{R}$, $\quad C(t) = C(-t)$,

(ii) *for each* $t, s \in \mathbb{R}$, *operators* $C(s), S(s), C(t), S(t)$ *commute,*

(iii) *for each* $x \in X$, *the mapping* $t \longrightarrow S(t)x$ *is continuous,*

(iv) $S(s+t) + S(s-t) = 2S(s)C(t)$,

(v) $S(s+t) = S(s)C(t) + S(t)C(s)$,

(vi) $S(t) = -S(-t)$,

(vii) *there exist constants* $M \geq 1$ *and* $\omega \geq 0$ *such that*

$$\text{for each } t \in \mathbb{R}: \quad \|C(t)\| \leq Me^{\omega|t|}, \tag{5}$$

$$\text{for each } s, t \in \mathbb{R}: \quad \|S(t) - S(s)\| \leq M \left| \int_s^t e^{\omega|r|} dr \right|. \tag{6}$$

THEOREM 2. ([**9**], Prop. 2.2.) *If the operator* A *is the infinitesimal generator of a cosine family* $\{C(t)\}_{t\in\mathbb{R}}$, *then*

(i) $D(A)$ *is a dense subspace of* X *and* A *is a closed operator,*

(ii) *if* $x \in X$ *and* $r, s \in \mathbb{R}$, *then* $z := \int_r^s S(t)xdt \in D(A)$ *and*

$$Az = C(s)x - C(r)x,$$

(iii) *if* $x \in X$ *then* $z := \int_0^s \int_0^r C(u)C(v)xdudv \in D(A)$ *and*

$$Az = 2^{-1}[C(s+r)x - C(s-r)x],$$

(iv) *if* $x \in X$, *then* $S(t)x \in E(A)$,

(v) *for* $x \in E(A)$: $S(t)x \in D(A)$ *and* $\frac{d^2}{dt^2}S(t)x = \frac{d}{dt}C(t)x = AS(t)x$,

(vi) *for* $x \in D(A)$: $C(t)x \in D(A)$ *and* $\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax$,

(vii) *if* $x \in D(A)$, *then* $S(t)x \in D(A)$ *and* $AS(t)x = S(t)Ax$,

(viii) $C(t+s) - C(t-s) = 2AS(t)S(s)$,

(ix) *if* M *and* ω *satisfy* (5), *then*

$$Re\lambda > \omega \Rightarrow \lambda^2 \in \varrho(A), \tag{7}$$

$$\lambda R(\lambda^2, A)x = \int_0^\infty e^{-\lambda t} C(t)xdt, \ \text{for } x \in X, \tag{8}$$

$$R(\lambda^2, A)x = \int_0^\infty e^{-\lambda t} S(t)xdt, \ \text{for } x \in X, \tag{9}$$

$$\left\| \frac{d^k}{d\lambda^k} \left[\lambda(A - \lambda^2)^{-1}\right] \right\| \leq Mk!(Re\lambda - \omega)^{-(k+1)}, \ \text{for } k = 0, 1, 2 \ldots \tag{10}$$

2. Weak solution. We consider the following abstract Cauchy problem

$$u''(t) = Au(t) + f(t), \ \ t \in \mathbb{R}, \tag{11}$$

$$u(0) = x, \ \ u'(0) = y, \tag{12}$$

where A is a closed linear operator and f is an X-valued function on $\mathbb{R}$.

A function $u : \mathbb{R} \longrightarrow X$ is called a classical solution of problem (11)–(12) if u is twice continuously differentiable, $u(t) \in D(A)$ for $t \in \mathbb{R}$ and satisfies (11)–(12). When A is densely defined, u is called a weak solution of problem (11)–(12) if u is continuous and for every $v \in D(A^*)$ the functions $\langle u(\cdot), v\rangle$ and $\frac{d}{dt}\langle u(\cdot), v\rangle$ are absolutely continuous and satisfy

$$\begin{gathered} \frac{d^2}{dt^2}\langle u(t), v\rangle = \langle u(t), A^*v\rangle + \langle f(t), v\rangle, \ \ a.e.\ t \in \mathbb{R}, \\ u(0) = x, \qquad \frac{d}{dt}\langle u(t), v\rangle_{|t=0} = \langle y, v\rangle. \end{gathered} \tag{13}$$

It is known that if the operator A is the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$ and under some assumption on the function f, problem (11)–(12) has a unique classical solution (if it exists) given by

$$u(t) = C(t)x + S(t)y + \int_0^t S(t-s)f(s)ds \ \ t \in [0, T]. \tag{14}$$

S. Kanda in [**5**] proved that

THEOREM 3. *Let A be a densely defined linear operator and $f \in L^1(0, T; X)$. Then the following assertions are equivalent:*

(i) for each $x, y \in X$ there exists exactly one weak solution of problem (11)–(12)*,*

(ii) A generates a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$ and in this case a weak solution u is given by (14)*.*

We will present theorems on weak continuity with respect to a parameter and weak differentiability with respect to a parameter of a weak solution of problem (11)–(12). Before proving our main theorems, we first state some facts about cosine and sine families.

THEOREM 4. *Let $\{C(t)\}_{t\in\mathbb{R}}$ be the cosine family with infinitesimal generator A. For each $t \in \mathbb{R}$, $x \in X$ i $v \in D(A^*)$ it follows that*

(i) $C^*(t)v \in D(A^*)$,

(ii) $S^*(t)v \in D(A^*)$,

(iii) $\frac{d}{dt}\langle C(t)x, v\rangle = \langle S(t)x, A^*v\rangle$,

(iv) $\frac{d}{dt}\langle S(t)x, v\rangle = \langle C(t)x, v\rangle$ *for* $v \in X^*$,

(v) $\frac{d^2}{dt^2}\langle C(t)x, v\rangle = \langle C(t)x, A^*v\rangle$,

(vi) $\frac{d^2}{dt^2}\langle S(t)x, v\rangle = \langle S(t)x, A^*v\rangle$.

Proof. To prove (i), we shall show that there exists a constant M such that for all $x \in D(A)$

$$|\langle Ax, C^*(t)v\rangle| \leq M\|x\|.$$

By Theorem 2 (vi) and (5),

$$|\langle Ax, C^*(t)v\rangle| = |\langle AC(t)x, v\rangle| = |\langle C(t)x, A^*v\rangle| \leq \|A^*v\|\,\|C(t)\|\,\|x\|\,.$$

Analogously one can prove assertion (ii).

Assertion (iii) follows from Theorem 2 (v) for $x \in D(A)$. For fixed $x \in X$ there exists a sequence $(x_n) \subset D(A)$ such that $x_n \to x$ and by continuity

$$\langle C(t)x_n, v\rangle \to \langle C(t)x, v\rangle, \tag{15}$$

when $n \to \infty$. Then

$$\left|\frac{d}{dt}\langle C(t)x_n, v\rangle - \langle S(t)x, A^*v\rangle\right| = |\langle S(t)x_n - S(t)x, A^*v\rangle| \leq M\,\|A^*v\|\,\|x_n - x\|\,,$$

where $M := \sup\{\|S(r)\|\,;\, r \in K\}$, and K is a compact subset of $\mathbb{R}$, $t \in K$. Hence,

$$\lim_{n\to\infty}\frac{d}{dt}\langle C(t)x_n, v\rangle = \langle S(t)x, A^*v\rangle\,, \tag{16}$$

uniformly in $t \in [a, b]$, $a < b$.

By (15) and (16), it follows that assertion (iii) holds for each $x \in X$.

By the density of $D(A)$, it is easy to prove (iv)–(vi). □

Theorem 5. *If the operators A and B generate strongly continuous cosine families $\{C_A(t)\}$ and $\{C_B(t)\}$, respectively, and $D(A^*) = D(B^*)$, then*

$$\langle S_A(t)x - S_B(t)x, v\rangle = \int_0^t \langle S_A(s)x, (A^* - B^*)S_B^*(t-s)v\rangle ds, \tag{17}$$

$$\langle C_A(t)x - C_B(t)x, v\rangle = \int_0^t \langle S_A(s)x, (A^* - B^*)C_B^*(t-s)v\rangle ds \tag{18}$$

for $v \in D(A^*) = D(B^*)$.

Proof. Let

$$g_1 : [0, t] \ni s \to \langle C_A(s)x, S_B^*(t-s)v\rangle + \langle S_A(s)x, C_B^*(t-s)v\rangle \in \mathbb{R},$$
$$g_2 : [0, t] \ni s \to \langle C_A(s)x, C_B^*(t-s)v\rangle + \langle S_A(s)x, S_B^*(t-s)B^*v\rangle \in \mathbb{R},$$

where $x \in X$ and $v \in D(A^*)$. By Theorem 4, we see that

$$\begin{aligned}\frac{d}{ds}g_1(s) &= \langle S_A(s)x, A^*S_B^*(t-s)v\rangle - \langle C_A(s)x, C_B^*(t-s)v\rangle \\ &\quad + \langle C_A(s)x, C_B^*(t-s)v\rangle - \langle S_A(s)x, S_B^*(t-s)B^*v\rangle \\ &= \langle S_A(s)x, (A^* - B^*)S_B^*(t-s)v\rangle.\end{aligned} \tag{19}$$

Analogously

$$\frac{d}{ds}g_2(s) = \langle S_A(s)x, (A^* - B^*)C_B^*(t-s)v\rangle. \tag{20}$$

By integrating (19) and (20) over $[0,t]$, we get (17) and (18). □

3. Dependence on a parameter. Let Ω be a compact subset of $\mathbb{R}^m$. We investigate the following Cauchy problem

$$\frac{d^2}{dt^2}u(t) = A_h u(t) + f_h(t) \;\; t \in (0,T], \;\; h \in \Omega \tag{21}$$

$$u(0) = x_h, \quad u'(0) = y_h. \tag{22}$$

In this section, we will need the following assumptions

Assumption (A). Assume that

1. for each $h \in \Omega$ an operator A_h generates a strongly continuous cosine family $\{C_h(t)\}$ satisfying (5) with constants M and ω not depending on h,
2. for each $h \in \Omega$: $0 \in \varrho(A_h)$,
3. domains $D(A_h^*) =: D^*$ do not depend on h,
4. for each $h_0 \in \Omega$ the mapping
$$\Omega \ni h \longrightarrow \overline{A_{h_0}^{-1}A_h} \in Aut(X)$$
is continuous at h_0.

Note that by Assumption (A), $A_{h_0}^{-1}A_h$ (see [**3, 4, 9**]) is a properly defined bounded operator. By the density of $D(A_h)$, one can easily verify that $\overline{A_{h_0}^{-1}A_h} \in Aut(X)$.

Assumption (B). Assume that for all $h_0 \in \Omega$ the family $\left\{\overline{A_{h_0}^{-1}A_h}\right\}_{h\in\Omega}$ has a weakly continuous weak derivative, i.e., there exists a weakly continuous family $\{\frac{\partial}{\partial h}\overline{A_{h_0}^{-1}A_h}\}_{h\in\Omega}$ of linear operators such that

$$\forall x \in X \;\forall v \in X^* : \quad \frac{\partial}{\partial h}\left\langle \overline{A_{h_0}^{-1}A_h}x, v\right\rangle = \left\langle \frac{\partial}{\partial h}\overline{A_{h_0}^{-1}A_h}x, v\right\rangle.$$

If the family $\{\overline{A_{h_0}^{-1}A_h}\}_{h\in\Omega}$ satisfies Assumptions (A) and (B), then it has the following properties

Theorem 6. ([**4**], *Th. 4.*)

(i) $\forall h_0 \in \Omega \quad \exists C > 0 \quad \forall h \in \Omega \quad h \neq h_0 :$

$$\left\| \frac{\overline{A_{h_0}^{-1}A_h} - I}{h - h_0} \right\| \leq C,$$

(ii) $\forall h, h_0 \in \Omega$ *the linear operator* $\frac{\partial}{\partial h}\overline{A_{h_0}^{-1}A_h}$ *is bounded,*

(iii) family $\{(\overline{A_{h_0}^{-1}A_h})^*\}_{h\in\Omega}$ *is* w^**-differentiable and*

$$\frac{\partial}{\partial h}\left[\overline{A_{h_0}^{-1}A_h}\right]^* = \left[\frac{\partial}{\partial h}\overline{A_{h_0}^{-1}A_h}\right]^*,$$

(iv) family $\{\overline{A_h^{-1}A_{h_0}}\}_{h\in\Omega}$ *has a weakly continuous weak derivative,*

(v) family $\{A_h^*\}_{h\in\Omega}$ *is* w^**-differentiable.*

Theorem 7. *If Assumption (A) holds, then for each* $v \in D^*$

$$\lim_{h\to h_0} \langle C_h(t)x, v\rangle = \langle C_{h_0}(t)x, v\rangle, \tag{23}$$

$$\lim_{h\to h_0} \langle S_h(t)x, v\rangle = \langle S_{h_0}(t)x, v\rangle, \tag{24}$$

uniformly in $(t,x) \in [0,T]\times K$*, where* K *is a compact subset of* X*.*

Proof. By (17), there is

$$\begin{aligned}\langle S_h(t)x - S_{h_0}(t)x, v\rangle &= \int_0^t \left\langle S_h(s)x, (A_h^* - A_{h_0}^*)S_{h_0}^*(t-s)v\right\rangle ds\\ &= \int_0^t \left\langle S_h(s)x, \left(\overline{A_{h_0}^{-1}A_h} - I\right)^* A_{h_0}^* S_{h_0}^*(t-s)v\right\rangle ds.\end{aligned}$$

From $\lim_{h\to h_0}\left[\overline{A_{h_0}^{-1}A_h} - I\right] = 0$ it follows that there exists a constant $M > 0$ such that $\left\|\overline{A_{h_0}^{-1}A_h} - I\right\| \le M$ for $h \in \Omega$.

Note that

$$\|A_{h_0}^* S_{h_0}^*(t-s)v\| = \|S_{h_0}^*(t-s)A_{h_0}^*v\| \le \|S_{h_0}(t-s)\|\|A_{h_0}^*v\|.$$

Therefore, by the Lebesgue theorem, the theorem is proved. □

Corollary 1. *If Assumption (A) holds in a reflexive Banach space, then for each* $v \in X^*$

$$\lim_{h\to h_0} \langle C_h(t)x, v\rangle = \langle C_{h_0}(t)x, v\rangle, \tag{25}$$

$$\lim_{h\to h_0} \langle S_h(t)x, v\rangle = \langle S_{h_0}(t)x, v\rangle, \tag{26}$$

uniformly in $(t,x) \in [0,T]\times K$*, where* K *is a compact subset of* X*.*

It is easy to prove the following theorem

Theorem 8. *If Assumption (A) holds and the mappings* $\Omega \ni h \to x_h \in X$*,* $\Omega \ni h \to y_h \in X$*,* $\Omega \ni h \to f_h \in L^1(0,T;X)$ *are continuous, then*

$$\lim_{h\to h_0} \langle u_h(t), v\rangle = \langle u_{h_0}(t), v\rangle$$

uniformly in $t \in [0,T]$, *where* $v \in D^*$, *and* u_h *is a weak solution of problem* (21)–(22).

Corollary 2. *If the family* $\{A_h\}_{h\in\Omega}$ *satisfies assumptions of Theorem 8 and* X *is a reflexive Banach space, then*

$$\forall v \in X^* : \quad \lim_{h\to h_0} \langle u_h(t), v\rangle = \langle u_{h_0}(t), v\rangle$$

uniformly in $t \in [0,T]$.

4. Differentiability of the weak solution with respect to a parameter.

Lemma 1. *Let* Ω *be a compact subset of* $\mathbb{R}$. *If* u_h *is a weak solution of problem* (21)–(22) *and* $v \in D^*$, *then*

$$\begin{aligned}\left\langle \frac{u_h(t)-u_{h_0}(t)}{h-h_0}, v\right\rangle &= \left\langle C_h(t)\frac{x_h-x_{h_0}}{h-h_0}, v\right\rangle + \left\langle S_h(t)\frac{y_h-y_{h_0}}{h-h_0}, v\right\rangle\\ &+\int_0^t \left\langle S_h(t-s)\frac{f_h(s)-f_{h_0}(s)}{h-h_0}, v\right\rangle ds\\ &+\int_0^t \left\langle S_h(s)x_{h_0}, \frac{A_h^*-A_{h_0}^*}{h-h_0}C_{h_0}^*(t-s)v\right\rangle ds\\ &+\int_0^t \left\langle S_h(s)y_{h_0}, \frac{A_h^*-A_{h_0}^*}{h-h_0}S_{h_0}^*(t-s)v\right\rangle ds\\ &+\int_0^t\int_s^t \left\langle S_h(r-s)f_{h_0}(s), \frac{A_h^*-A_{h_0}^*}{h-h_0}S_{h_0}^*(t-r)v\right\rangle drds.\end{aligned} \tag{27}$$

Proof. For a fixed $v \in D^*$, there is

$$\langle C_h(t)x_h - C_{h_0}(t)x_{h_0}, v\rangle = \langle [C_h(t)-C_{h_0}(t)]x_{h_0} + C_h(t)(x_h-x_{h_0}), v\rangle.$$

Therefore, by Theorem 5,

$$\begin{aligned}\langle C_h(t)x_h - C_{h_0}(t)x_{h_0}, v\rangle &= \langle C_h(t)(x_h-x_{h_0}), v\rangle\\ &+\int_0^t \left\langle S_h(s)x_{h_0}, [A_h^*-A_{h_0}^*]C_{h_0}^*(t-s)v\right\rangle ds.\end{aligned} \tag{28}$$

Analogously

$$\begin{aligned}\langle S_h(t)y_h - S_{h_0}(t)y_{h_0}, v\rangle &= \langle S_h(t)(y_h-y_{h_0}), v\rangle\\ &+\int_0^t \langle S_h(s)y_{h_0}, [A_h^*-A_{h_0}^*]S_{h_0}^*(t-s)v\rangle ds.\end{aligned} \tag{29}$$

By (17), we obtain

$$\begin{aligned}
&\int_0^t \langle [S_h(t-s) - S_{h_0}(t-s)] f_{h_0}(s), v\rangle ds \\
&= \int_0^t \int_0^{t-s} \langle S_h(\tau) f_{h_0}(s), [A_h^* - A_{h_0}^*] S_{h_0}^*(t-s-\tau)v\rangle d\tau ds \\
&\qquad\qquad \tau + s =: r \\
&= \int_0^t \int_s^t \langle S_h(r-s) f_{h_0}(s), [A_h^* - A_{h_0}^*] S_{h_0}^*(t-r)v\rangle dr ds.
\end{aligned} \tag{30}$$

Functions u_h and u_{h_0} given by

$$u_h(t) = C_h(t)x_h + S_h(t)y_h + \int_0^t S_h(t-s)f_h(s)ds,$$

$$u_{h_0}(t) = C_{h_0}(t)x_{h_0} + S_{h_0}(t)y_{h_0} + \int_0^t S_{h_0}(t-s)f_{h_0}(s)ds$$

are weak solutions of problem (21)–(22) with parameter h and h_0, respectively. From (28), (29) and (30) it follows that $\left\langle \frac{u_h(t) - u_{h_0}(t)}{h - h_0}, v \right\rangle$ satisfies (27). $\square$

Now suppose that X is a reflexive Banach space.

THEOREM 9. *Let Ω be a compact subset of $\mathbb{R}$. Let the family $\{A_h\}_{h\in\Omega}$ satisfy Assumptions (A) and (B) in the reflexive Banach space X. If the mappings $\Omega \ni h \to x_h \in X$, $\Omega \ni h \to y_h \in X$ and $f : \Omega \ni h \to f_h \in L^1(0,T;X)$ are of C^1 class, then the mapping*

$$u : \Omega \times [0,T] \ni (h,t) \longrightarrow u_h(t) \in X,$$

where $u_h(\cdot)$ is a weak solution of problem (21)–(22), is weakly differentiable on D^* with respect to h, i.e., for each $v \in D^*$, the function

$$(h,t) \to \langle u_h(t), v\rangle$$

is differentiable with respect to h and

$$\begin{aligned}
\frac{\partial}{\partial h}\langle u_h(t), v\rangle_{|h=h_0} &= \left\langle C_{h_0}(t)x'_{h_0}, v\right\rangle + \left\langle S_{h_0}(t)y'_{h_0}, v\right\rangle \\
&+ \int_0^t \left\langle S_{h_0}(t-s)f'_{h_0}(s), v\right\rangle ds \\
&+ \int_0^t \left\langle S_{h_0}(s)x_{h_0}, \left(A_{h_0}^*\right)' C_{h_0}^*(t-s)v\right\rangle ds \\
&+ \int_0^t \left\langle S_{h_0}(s)y_{h_0}, \left(A_{h_0}^*\right)' S_{h_0}^*(t-s)v\right\rangle ds \\
&+ \int_0^t \int_s^t \left\langle S_{h_0}(r-s)f_{h_0}(s), \left(A_{h_0}^*\right)' S_{h_0}^*(t-r)v\right\rangle dr ds,
\end{aligned}$$

where the symbol "$'$" denotes differentiation with respect to h and, moreover,

$$\left(A_{h_0}^*\right)' := \left(\frac{\partial}{\partial h}\overline{A_{h_0}^{-1}A_h}_{|h=h_0}\right)^* A_{h_0}^*.$$

Proof. Let us consider $\left\langle \frac{u_h(t)-u_{h_0}(t)}{h-h_0}, v\right\rangle$ given by (27). By (5) and (6),

$$\lim_{h\to h_0}\left\langle C_h(t)\frac{x_h - x_{h_0}}{h-h_0}, v\right\rangle = \left\langle C_{h_0}(t)x_{h_0}', v\right\rangle,$$

$$\lim_{h\to h_0}\left\langle S_h(t)\frac{y_h - y_{h_0}}{h-h_0}, v\right\rangle = \left\langle S_{h_0}(t)y_{h_0}', v\right\rangle,$$

$$\lim_{h\to h_0}\int_0^t \left\langle S_h(t-s)\frac{f_h(s) - f_{h_0}(s)}{h-h_0}, v\right\rangle ds = \int_0^t \left\langle S_{h_0}(t-s)f_{h_0}'(s), v\right\rangle ds,$$

uniformly in $t \in [0,T]$.
By Theorem 6,

$$\frac{A_h^* - A_{h_0}^*}{h-h_0} = \left(\frac{\overline{A_{h_0}^{-1}A_h} - I}{h-h_0}\right)^* A_{h_0}^*,$$

and

$$\left\|\frac{\overline{A_{h_0}^{-1}A_h} - I}{h-h_0}\right\| \le C_1 \quad \text{for } h \in \Omega.$$

From (5) and (6) it follows that for each $t \in [0,T]$

$$\left\|A_{h_0}^* C_{h_0}^*(t-s)v\right\| \le Me^{\omega T}\left\|A_{h_0}^* v\right\|,$$
$$\left\|A_{h_0}^* S_{h_0}^*(t-s)v\right\| \le M\omega^{-1}(e^{\omega T}-1)\left\|A_{h_0}^* v\right\|,$$

therefore, by the Lebesgue theorem,

$$\lim_{h\to h_0}\int_0^t \left\langle S_h(s)x_{h_0}, \frac{A_h^* - A_{h_0}^*}{h-h_0}C_{h_0}^*(t-s)v\right\rangle ds$$
$$= \int_0^t \left\langle S_{h_0}(s)x_{h_0}, \left(A_{h_0}^*\right)' C_{h_0}^*(t-s)v\right\rangle ds,$$

and

$$\lim_{h\to h_0}\int_0^t \left\langle S_h(s)y_{h_0}, \frac{A_h^* - A_{h_0}^*}{h-h_0}S_{h_0}^*(t-s)v\right\rangle ds$$
$$= \int_0^t \left\langle S_{h_0}(s)y_{h_0}, \left(A_{h_0}^*\right)' S_{h_0}^*(t-s)v\right\rangle ds,$$

uniformly in $t \in [0, T]$. Moreover,

$$\left| \left\langle S_h(r-s) f_{h_0}(s), \frac{A_h^* - A_{h_0}^*}{h - h_0} S_{h_0}^*(t-r) v \right\rangle \right|$$
$$\leq \left\| \frac{A_h^* - A_{h_0}^*}{h - h_0} S_{h_0}^*(t-r) v \right\| \left\| S_h(r-s) \right\| \left\| f_{h_0}(s) \right\| \leq C \left\| f_{h_0}(s) \right\|.$$

By the Lebesgue theorem,

$$\lim_{h \to h_0} \int_0^t \int_s^t \left\langle S_h(r-s) f_{h_0}(s), \frac{A_h^* - A_{h_0}^*}{h - h_0} S_{h_0}^*(t-r) v \right\rangle drds$$
$$= \int_0^t \int_s^t \left\langle S_{h_0}(r-s) f_{h_0}(s), \left(A_{h_0}^*\right)' S_{h_0}^*(t-r) v \right\rangle drds.$$

This ends the proof. □

References

1. Fattorini H. O., *Second Order Linear Differential Equations in Banach Spaces*, North-Holland, 1985.
2. Hill E., Philips R. S., *Functional Analysis and Semi-Groups*, 1985.
3. Juzyniec M., *Dependence of a weak solution of the first order differential equation on a parameter*, Univ. Iagel. Acta Math., **XLVI** (2008), 35–46.
4. Juzyniec M., *Weak fundamental solution of the first order evolution equation*, Univ. Iagel. Acta Math., **XLVII** (2009), 249–264.
5. Kanda S., *Cosine families and weak solutions of second order differential equations*, Proc. Japan Acad., **54** (1978), 119–123.
6. Kato T., *Perturbation theory for linear operators*, Springer-Verlag, 1966.
7. Krein S. G., *Linear differential equations in Banach spaces*, Transl. Math. Monographs, Vol. **29**, Amer. Math. Soc., 1971.
8. Pazy A., *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, 1983.
9. Travis C. C., Webb G. F., *Cosine families and abstract nonlinear second order differential equations*, Acta Math. Sci. Hung., **32** (1978), 75–96.
10. Winiarska T., Winiarski T., *Regularity of domains of parameterized families of closed linear operators*, Ann. Polon. Math., **80** (2003), 231–241.

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Institute of Mathematics
Cracow University of Technology
Warszawska 24
31-155 Kraków, Poland
e-mail: juzyniec@usk.pk.edu.pl